Handout 2

1 Optimization with equality constraints

The Lagrange method combines the objective function and constraints into a single equation:

$$\mathcal{L}(x_1, x_2, ..., x_n) = (\text{objective function}) + \lambda_1 \,(\text{first constraint}) + \cdots + \lambda_n \left(n^{th} \text{ constraint}\right) \quad (1)$$

Calculate first order condition with respect to $x_1, x_2, ..., x_n, \lambda_1, ..., \lambda_n$. Then solve the simultaneous equations to obtain $x_1^*, x_2^*, ..., x_n^*$.

2 Percentage change, rate of change and growth rate

- **Percentage change** of $X = \dfrac{X' - X}{X}$.
- **Rate of change** of $X =$ **growth** of $X = \dot{X} = \dfrac{dX}{dt} = \dfrac{X' - X}{t' - t} =$ change of X in a small time period.
- $\dfrac{d \ln X(t)}{dt} = \dfrac{d \ln X(t)}{dX(t)} \cdot \dfrac{dX(t)}{dt} = \dfrac{1}{X(t)} \dot{X}(t) = \dfrac{\dot{X}}{X}$.
- **Growth rate** of $X =$ proportional rate of change of $X = \dfrac{\dot{X}}{X} = \dfrac{\frac{X'-X}{t'-t}}{X} = \dfrac{\frac{X'-X}{X}}{t'-t} =$ percentage change of X in a small time period ($\frac{\dot{X}}{X} \times 100 = \%\Delta X =$ percentage growth rate of X).
- *Growth rate of a variable = rate of change of its natural log.*
  - Suppose y's percentage change is $C\%$, then we can write:
  $$y' = (1 + C\%)y$$
  $$\Longrightarrow \ln(y') - \ln(y) = \ln(1 + C\%) \approx C\%.$$
  - Divide both sides by a small time period: $\dfrac{\ln(y') - \ln(y)}{t' - t} = \dfrac{C\%}{t' - t}$.
  - Left hand side: rate of change of y's natural log $\Longrightarrow \dfrac{d \ln y}{dt}$.
  - Right hand side: growth rate of y .
- Application: the quantity equation of money.
  - $\boxed{M \times V = P \times Y}$
  - Growth rate derivation:
  $$\begin{aligned} \ln(MV) = \ln(PY) &\Longrightarrow \ln M + \ln V = \ln P + \ln Y \\ &\Longrightarrow \frac{\partial \ln M}{\partial t} + \frac{\partial \ln V}{\partial t} = \frac{\partial \ln P}{\partial t} + \frac{\partial \ln Y}{\partial t} \\ &\Longrightarrow \frac{\dot{M}}{M} + \frac{\dot{V}}{V} = \frac{\dot{P}}{P} + \frac{\dot{Y}}{Y} \\ &\Longrightarrow \boxed{\%\boldsymbol{\Delta M} + \%\boldsymbol{\Delta V} = \%\boldsymbol{\Delta P} + \%\boldsymbol{\Delta Y}} \end{aligned}$$
  - $\%\Delta X$: **percentage change of $\boldsymbol{X}$** in a small time period (e.g. X grows at rate $\Delta\%$).
  - Since V is constant over time, $\%\Delta V$ is 0 (no growth) $\leftarrow$ This is a very strong assumption.

3 Exercises

1. Suppose real GDP grows at rate 3% per year. At what rate of growth must the money supply expand for the inflation rate to be equal to 2%?

2. Consider an Economy where the representative household is endowed with capital K has a total of h hours in a day and has preferences over consumption c and leisure ℓ such that we can model with the following utility function:
$$u(C,\ell) = C - \frac{(h-\ell)^2}{2}.$$
The representative firm in the Economy has the following production function:
$$Y = AK^{\alpha}L^{1-\alpha}.$$
Suppose household do not invest and there is no government spending.
   (a) Write down the household's optimization problem.
   (b) Find the household's labor supply.
   (c) What is the effect of real wage w to labor supply?
   (d) Solve for the equilibrium levels of output, labor, and consumption.

3. Consider a representative household with preferences over consumption C and leisure ℓ given by:
$$u(C,\ell) = \frac{C^{1-\gamma}}{1-\gamma} + \phi\ell$$
where γ and ϕ are constants and strictly positive. Suppose there is no capital and the individual is endowed with h hours of time.
   (a) Solve for her optimal choices of consumption and leisure.
   (b) Under what conditions does her labor supply curve slope up or down?
   (c) What is the relationship between γ and the *income effect* and the *substitution effect*?

4. Consider an economy with government involve. A representative worker has preferences over consumption C and leisure ℓ given by:
$$u(C,\ell) = \log(C) + \phi\ell$$
where ϕ is a constant and with budget constraint:
$$C = w(1-\ell) - T$$
where $1-\ell$ is labor supply and T indicates lump sum taxes. The government spending G is a fixed proportion g of output: $G = gY$. Suppose productivity is fixed at one unit, and there is a representative firm who produces according to:
$$Y = L.$$
   (a) Find the equilibrium values of employment, output, consumption, and the real wage.
   (b) Suppose that the government increases its spending as a fraction of output from g to g'. What are the equilibrium effects on output, labor, consumption, and the real wage?

4 Appendix: Elasticity

An application of "percentage change", "growth", and "growth rate" is elasticity in Microeconomics. Elasticity of y with respect ot x refers to the percentage change in y induced by a (small) percentage change in x:

$$\begin{aligned}\epsilon &= \frac{d\ln y}{d\ln x} = \frac{\frac{d\ln y(t)}{dt}}{\frac{d\ln x(t)}{dt}} = \frac{\frac{d\ln y(t)}{dt}\times 100}{\frac{d\ln x(t)}{dt}\times 100} = \frac{\%\Delta y}{\%\Delta x} = \frac{\frac{\dot{y}}{y}}{\frac{\dot{x}}{x}} = \frac{\frac{\frac{dy}{dt}}{y}}{\frac{\frac{dx}{dt}}{x}} = \frac{\frac{\frac{y'-y}{y}}{t'-t}}{\frac{\frac{x'-x}{x}}{t'-t}} = \frac{\frac{y'-y}{y}}{\frac{x'-x}{x}} \\ &= \frac{\text{(percentage) growth rate of } y}{\text{(percentage) growth rate of } x} = \frac{\text{percentage change of } y}{\text{percentage change of } x} \\ &= \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx}\frac{x}{y}.\end{aligned} \tag{2}$$