Handout 3

1 Growth review

- Suppose X_t is a variable that changes over time t. e.g. output, capital.
- How to calculate the growth rate of X_t?
  1. Take the natural log.
  2. Differentiate with respect to time t.
- $\frac{d\ln X_t}{dt} = \frac{d\ln X_t}{dX_t} \cdot \frac{dX_t}{dt} = \frac{1}{X_t} \cdot \dot{X}_t = \boxed{\frac{\dot{X}_t}{X_t} = g_{X_t} = \text{growth rate of } X_t}$ where $\boxed{\dot{X}_t = \frac{dX_t}{dt}}$.
- Growth rate of X = percentage change of X in a small period of time.
- e.g. What is the growth equation of $z_t = a \cdot x_t^b \cdot y_t^c$ where a is a constant?

$$\begin{aligned}
&\ln z_t = \ln a + b\ln x_t + c\ln y_t \\
&\Longrightarrow \frac{d\ln z_t}{dt} = \underbrace{\frac{d\ln a}{dt}}_{=0} + b \cdot \frac{d\ln x_t}{dt} + c \cdot \frac{d\ln y_t}{dt} \\
&\Longrightarrow \frac{\dot{z}_t}{z_t} = b \cdot \frac{\dot{x}_t}{x_t} + c \cdot \frac{\dot{y}_t}{y_t} \\
&\Longrightarrow g_{z_t} = b \cdot g_{x_t} + c \cdot g_{y_t}.
\end{aligned}$$

2 The Solow growth model (with 4 components in continuous time)

- Introduction:
  1. Builds on the production model.
  2. Adds capital accumulation.
  3. developed in the 1950s by Robert Solow (Nobel prize winner in 1987).
  4. In the real world, we observe different time series variables, e.g. GDP, employment. From these data, we find some patterns *over time* or *cross-section*.
  5. A fact about the pattern : richer countries tend to have more capital.
  6. We want to understand what lies beneath these patterns and learn some lessons that help us to make decisions and establish policy.
  7. How to do it? We will construct models and use their mathematical features to explain the patterns. e.g. use the Solow model to explain how the variables grow over time.
- **Growth of input factors**:
  1. Knowledge (E_t) grows at rate: $g \Longrightarrow \boxed{g = \frac{\dot{E}_t}{E_t}}$
  2. Labor (L_t) grows at rate: $n \Longrightarrow \boxed{n = \frac{\dot{L}_t}{L_t}}$
- Goods: **consume** or **invest/save** (more capital→ more output in the future).
- Capital: **depreciates** every period (only δ portion remains).

2.1 The production function

- Cobb-Douglas, CRS in capital and **effective** labor: $Y_t = A \cdot F(K_t, E_t L_t) = A \cdot K_t^{\alpha} (E_t L_t)^{1-\alpha}$.
- Change the production function into a **intensive form (per-capita terms)** production function (y_t):
  - $$\frac{Y_t}{E_t L_t} = \frac{1}{E_t L_t} A \cdot F(K_t, E_t L_t) = A \cdot F\left(\frac{K_t}{E_t L_t}, \frac{E_t L_t}{E_t L_t}\right)$$
    $$= A \cdot F\left(\frac{K_t}{E_t L_t}, 1\right) \equiv A \cdot f(k_t, 1) = y_t \Longrightarrow \boxed{y_t = \frac{Y_t}{E_t L_t}}$$
  - Why? ∵ interested in how capital per-capita affects output. We don't explain population growth.
  - Steps:
    1. Define $\boxed{k_t = \frac{K_t}{E_t L_t}}$ where E_t describes the **effectiveness** of worker. Hence, k_t denotes **capital per effective worker**.
    2. Divide the production function (Y_t) by $E_t L_t$:

$$y_t = \frac{Y_t}{E_t L_t} = \frac{A \cdot K_t^{\alpha} (E_t L_t)^{1-\alpha}}{E_t L_t}$$
$$= A \cdot \frac{K_t^{\alpha}}{(E_t L_t)^{\alpha}} \cdot \frac{(E_t L_t)^{1-\alpha}}{(E_t L_t)^{1-\alpha}}$$
$$= A \cdot \left(\frac{K_t}{E_t L_t}\right)^{\alpha} \cdot \left(\frac{E_t L_t}{E_t L_t}\right)^{1-\alpha}$$
$$= A \cdot k_t^{\alpha}.$$

$\Longrightarrow$ So we obtain the per-capita production function: $\boxed{y_t = A k_t^{\alpha}.}$

2.2 The resource constraint

Notice that the goods (output, Y) can either be consumed or saved for investing. So at optimal, we can write down the following relationship: $Y_t = C_t + I_t$.
Similarly, we change the equation into intensive form by divide the entire equation by $E_t L_t$:

$$\underbrace{\frac{Y_t}{E_t L_t}}_{y_t} = \frac{C_t}{E_t L_t} + \frac{I_t}{E_t L_t}$$
$$= c_t + i_t$$

$\Longrightarrow$ The per-capita resource constraint: $\boxed{y_t = c_t + i_t.}$

2.3 The consumption and saving behavior

- Recall that when goods are produced, **the part that is not saved** by the household **is consumed** by the household. What we mean saving here is the that the household will save a portion of goods in order to invest it into the next period's production. Notice that the saving rate (s) is exogenous.
- You can think of it as we harvest corn, so corn is our goods. And we will use corn as fertilizer (invest corn into next period's production) to produce more corns in the future. So we can write

$$C_t = (1-s)Y_t \xrightarrow{\text{write in per-capita terms}} c_t \equiv \frac{C_t}{E_t L_t} = (1-s)\frac{Y_t}{E_t L_t} = (1-s)y_t \Longrightarrow \boxed{c_t = (1-s)y_t.}$$

2.4 The law of motion for capital

- The existing capital depreciates over time at a fixed rate δ where $0 \leq \delta \leq 1$. So the capital stock in the beginning of the **next** period equals to **this** period's existing depreciated capital + **this** period's investments:

$$\underbrace{K_{t+1}}_{\text{next period's capital}} = \underbrace{(1-\delta)K_t}_{\text{depreciated capital of current period}} + \underbrace{I_t}_{\text{investments of current period}} \tag{1}$$

Since **investment** is the **goods that you save**, we have

$$I_t = s \cdot Y_t \tag{2}$$

So we can rewrite equation (1) as the following:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &= (1-\delta)K_t + sY_t \\ \implies (K_{t+1} - K_t) &= sY_t - \delta K_t \end{aligned} \tag{3}$$

- Next, from section one, recall we define $\dot{X}$ as $\frac{dX}{dt}$, a change in X in a small period of time. If we apply the same concept here then $(K_{t+1} - K_t)$ is the change in capital level between time t and $t+1$.
- If this time period is very small, then $(K_{t+1} - K_t) \approx \dot{K}_t$. Hence, we can write equation (3) as

$$\boxed{\dot{K}_t = sY_t - \delta K_t.} \tag{4}$$

- Recall in section 2.1 when we derive the per-capita production function, in step 1, we define $\boxed{k_t = \frac{K_t}{E_t L_t}.}$

1. Take natural log of both sides:

$$\begin{aligned} \ln k_t = \ln\left(\frac{K_t}{E_t L_t}\right) &\implies \ln K_t - \ln(E_t L_t) = \ln K_t - \ln E_t - \ln L_t \\ &\implies \boldsymbol{\ln k_t = \ln K_t - \ln E_t - \ln L_t} \end{aligned}$$

2. Differentiate both sides with respect to time and substitute $\dot{K}_t$ with equation (4). Furthermore, recall in page 1 we have knowledge E_t grows at rate g and labor L_t grows at rate n. So we have the following:

$$\begin{aligned} \frac{d\ln k_t}{dt} &= \frac{d\ln K_t}{dt} - \frac{d\ln E_t}{dt} - \frac{d\ln L_t}{dt} \\ \implies \frac{\dot{k}_t}{k_t} &= \frac{\dot{K}_t}{K_t} - \frac{\dot{E}_t}{E_t} - \frac{\dot{L}_t}{L_t} \\ \implies \dot{k}_t &= \frac{\dot{K}_t}{K_t}k_t - \frac{\dot{E}_t}{E_t}k_t - \frac{\dot{L}_t}{L_t}k_t \\ &= \frac{sY_t - \delta K_t}{K_t}k_t - g \cdot k_t - n \cdot k_t \\ &= \frac{sY_t}{K_t} \cdot k_t - \delta k_t - gk_t - nk_t \\ &= \frac{sY_t}{K_t} \cdot \frac{K_t}{E_t L_t} - (\delta + g + n)\, k_t \\ &= s \cdot \frac{Y_t}{E_t L_t} - (\delta + g + n)\, k_t \\ &= s \cdot y_t - (\delta + g + n)\, k_t \end{aligned}$$

$\implies$ So we obtain the per-capita law of motion for capital: $\boxed{\dot{k}_t = sy_t - (\delta + g + n)\, k_t.}$

3 A summary of the transition equations (the boxed results)

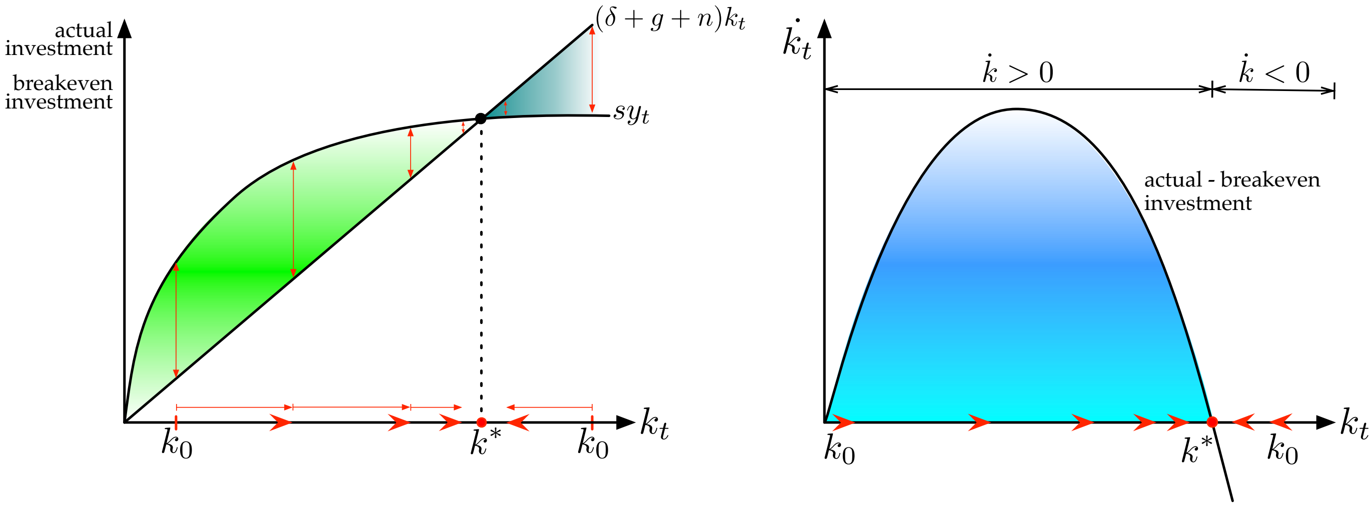
- Growth of input factors:
  - Knowledge E_t grows at rate g: $E_{t+1} = (1+g)E_t$
  - Labor L_t grows at rate n: $L_{t+1} = (1+n)L_t$
- Define intensive (or per-capita) form: $y = \frac{Y_t}{E_t L_t}$, $k_t = \frac{K_t}{E_t L_t}$, $c_t = \frac{C_t}{E_t L_t}$, $i_t = \frac{I_t}{E_t L_t}$.
- Per-capita production function: $y_t = Ak_t^\alpha$.
- Per-capita resource constraint: $y_t = c_t + i_t$.
- Per-capita consumption and saving behavior: $c_t = (1-s)y_t$.
- Per-capita law of motion for capital: $\dot{k}_t = sy_t - (\delta + g + n)k_t$.

4 The steady state equilibrium

- What is a steady state? If a system is in steady state, then it **remains at the same level over time**.
- If a variable X_t has a steady state, then $\dot{X}_t = 0$ and $X_t = X$ (can drop the time subscript), $\forall t$.
- From the Per-capita law of motion for capital:

$$\underbrace{\dot{k}_t}_{\text{rate of change (\textbf{growth}) of capital \textbf{stock} per unit of effective labor}} = \underbrace{sy_t}_{\text{\textbf{actual investment} per unit of effective labor}} - \underbrace{(\delta+g+n)k_t}_{\text{\textbf{breakeven investment} per unit of effective labor: amount of investment that must be done just to keep } k \text{ at its existing level}}$$

  - actual investment > breakeven investment $\Longrightarrow \dot{k}_t > 0$ ($k_t \uparrow$, then $\dot{k}_t$ decreases)
  - actual investment < breakeven investment $\Longrightarrow \dot{k}_t < 0$ ($k_t \downarrow$, then $\dot{k}_t$ increases)
  - The **phase diagram**:



  - k^* is where the actual investment equals the breakeven investment. At this point, $\dot{k}_t = 0$.
  - Regardless of what the initial capital stock level (k_0) is, it converges to k^* and remains there.
  - capital per-capita (per unit of efficient worker), k_t, has a steady state k^*. So k_t has no growth.

- To show it mathematically, starting from the per-capita law of motion for capital: $\dot{k}_t = sy_t - (\delta + g + n)k_t$
  * At steady state $\dot{k}_t = 0$, which yields

$$\begin{aligned} s \cdot y &= (\delta + g + n)k \\ &\implies s \cdot Ak^{\alpha} = (\delta + g + n)k \\ &\implies k^{1-\alpha} = \left(\frac{sA}{\delta + g + n}\right) \\ &\implies \boxed{k^* = \left(\frac{sA}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}} \end{aligned} \tag{5}$$

  * At steady state, from equation (5), $s, A, \delta, g, n, \alpha$ are just numbers (parameters) not variables, so capital per-capita k^* is a constant term (no growth).
  * $s \uparrow \rightarrow\ k^* \uparrow$: countries that have high saving rates will tend to be richer.
  * $n \uparrow \rightarrow\ k^* \downarrow$: countries that have high population growth rate tend to be poorer.

- From the capital per-capita definition:

$$\underbrace{k_t}_{\text{no growth}} = \frac{K_t}{\underbrace{E_t}_{\text{grow at rate } g} \underbrace{L_t}_{\text{grow at rate } n}} \tag{6}$$

  - Notice that when we calculate variable per-capita, we divide that variable by $E_t L_t$.
  - Hence, in this handout, **when we say "per-capital", we mean "per effective worker"**.
  - Notice that k_t doesn't grow. On the right hand side of equation (6), in the denominator, knowledge grows at rate g and labor (population) grows at rate n.
  - Focusing on equation (6), if the left hand side does not grow, the right hand side must also does not grow. In this case, it must be the numerator and the denominator grows at the same rate. Thus, the capital level must grows at the same rate as the denominator.
  - At steady state, the capital level K_t grows at rate $g + n$.

- Let's ignore the effect of knowledge. Instead of capital "per efficient worker," the **capital per worker** is the following:

$$\mathcal{K}_t = \frac{\overbrace{K_t}^{\text{grow at rate } g+n}}{\underbrace{L_t}_{\text{grow at rate } n}}$$

  - Notice that from section 1, the growth equation of $\dfrac{X}{Y}$ is $(g_X - g_Y)$.
  - $\mathcal{K}_t$ has growth rate of $g_{K_t} - g_{L_t} = (g + n) - n = g$.
  - At steady state, the capital per worker $\mathcal{K}_t$ grows at rate g.

- From the per-capita output function, at steady state:

$$y_t = \underbrace{A}_{\text{a number}} \cdot \underbrace{k_t}_{\text{no growth}}{}^{\alpha \leftarrow \text{a number}}$$

  - The only variable in the output transition equation (k_t) doesn't grow.
  - At steady state, the output per-capita y_t has no growth.

- From the output per-capita definition:

$$\underbrace{y_t}_{\text{no growth}} = \frac{Y_t}{\underbrace{E_t}_{\text{grow at rate } g} \underbrace{L_t}_{\text{grow at rate } n}} \tag{7}$$

  - Analogously, since y_t doesn't grow, on the right hand, the numerator and the denominator must have the same growth rate.
  - At steady state, the output level Y_t grows at rate $g+n$.

- Let's ignore the effect of knowledge again. Instead of output "per efficient worker," the **output per worker** is the following:

$$\mathcal{Y}_t = \frac{\overbrace{Y_t}^{\text{grow at rate } g+n}}{\underbrace{L_t}_{\text{grow at rate } n}}$$

  - $\mathcal{Y}_t$ has growth rate of $g_{Y_t} - g_{L_t} = (g+n) - n = g$.
  - At steady state, the output per worker $\mathcal{Y}_t$ grows at rate g.

- On the balance growth path (at steady state), the growth rate of
  1. capital per effective worker k_t is 0
  2. capital per worker $\mathcal{K}_t$ is g
  3. capital level K_t is $g+n$

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  4. output per effective worker k_t is 0
  5. output per worker $\mathcal{Y}_t$ is g
  6. output level Y_t is $g+n$

5 Quick notes

- We use **continuous time** representation $\dot{\boldsymbol{k}}_{\boldsymbol{t}} = \frac{dk_t}{dt}$ to denote *capital change in a period of time*. In order to avoid calculus, in lecture, instead we use **discrete time** representation $\boldsymbol{\Delta k_{t+1}} = k_{t+1} - k_t$.

- If we write the investment-saving (equation 2) in per-capita form[1] $i_t = s \cdot y_t$, then the law of motion for capital can be rewritten as the following:

$$\dot{k}_t = \boldsymbol{s y_t} - (\delta + g + n)k_t \implies \dot{k}_t = \boldsymbol{i_t} - (\delta + g + n)k_t. \tag{8}$$

- This handout deals with the same production function as in the lecture notes. However, the model we encountered in class does not incorporate the effect of knowledge: $Y_t = AF(K, L)$.

- In this case, we define the capital per-capita as $k_t = \frac{K_t}{L_t}$ $\implies$ $\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t}$.

- Since there is no knowledge in the setup in class, we can set $g = 0$.

- Also we temporarily assume that the population is constant (the growth rate of labor is 0) which is setting $n = 0$.

- Setting $g = 0$ and $n = 0$, all the results and graphs in this handout will be the same as what we've seen in class.

[1] Deriving the per-capita form of investment-saving equation: $I_t = s \cdot Y_t \implies \frac{I_t}{E_t L_t} = s \frac{Y_t}{E_t L_t} \implies i_t = s \cdot y_t$.

The household's optimization problem

$$\max_{C,L} \sqrt{C} + \phi\sqrt{1-L} \quad \text{s.t.} \quad PC = WL + RK$$

- Solution Method I: Substitution Approach

Define: $w = \frac{W}{P}$ and $r = \frac{R}{P}$.

$$\begin{aligned} &\max_{C,L} \sqrt{C} + \phi\sqrt{1-L} \quad \text{s.t.} \quad PC = WL + RK \\ &\Longrightarrow \max_{C,L} \sqrt{C} + \phi\sqrt{1-L} \quad \text{s.t.} \quad C = \frac{W}{P}L + \frac{R}{P}K = wL + rK \\ \text{Substitute } C \text{ with constraint:} \quad &\Longrightarrow \max_{L} \sqrt{wL + rK} + \phi\sqrt{1-L} \\ &\Longrightarrow \max_{L} (wL + rK)^{\frac{1}{2}} + \phi(1-L)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{F.O.C [L]:} \quad &\frac{1}{2}(wL + rK)^{-\frac{1}{2}} w + \frac{1}{2}\phi(1-L)^{-\frac{1}{2}}(-1) = 0 \\ &\Longrightarrow (wL + rK)^{-\frac{1}{2}} w = \phi(1-L)^{-\frac{1}{2}} \\ &\Longrightarrow (wL + rK) \cdot w^{-2} = \phi^{-2}(1-L) \\ &\Longrightarrow w^{-1}L + rKw^{-2} = \phi^{-2} - \phi^{-2}L \\ &\Longrightarrow \left(\phi^{-2} + w^{-1}\right) L = \phi^{-2} - rKw^{-2} \\ &\Longrightarrow L = \frac{\phi^{-2} - rKw^{-2}}{\phi^{-2} + w^{-1}} = \frac{\phi^{-2}w^2 - rK}{\phi^{-2}w^2 + w} = \frac{w^2 - rK\phi^2}{w^2 + w\phi^2} = \frac{w - \phi^2\left(\frac{r}{w}\right)K}{\phi^2 + w}. \end{aligned}$$

- Solution Method II: Lagrange Approach

Define: $r = \frac{R}{P}$ and $w = \frac{W}{P}$.

$$\max_{C,L} \sqrt{C} + \phi\sqrt{1-L} \quad \text{s.t.} \quad C = wL + rK$$

$$\mathcal{L} = C^{\frac{1}{2}} + \phi(1-L)^{\frac{1}{2}} + \lambda(C - wL - rK)$$

$$\frac{\partial \mathcal{L}}{\partial C} = \frac{1}{2}C^{-\frac{1}{2}} + \lambda = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \frac{1}{2}\phi(1-L)^{-\frac{1}{2}}(-1) - \lambda w = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C - wL - rK = 0 \tag{11}$$

$$\begin{aligned} \text{Divide (9) by (10):} \quad &\frac{C^{-\frac{1}{2}}}{\phi(1-L)^{-\frac{1}{2}}} = \frac{1}{w} \\ &\Longrightarrow \frac{C}{\phi^{-2}(1-L)} = \frac{1}{w^{-2}} \\ \text{Substitue } C \text{ with (11):} \quad &\Longrightarrow \frac{wL + rK}{\phi^{-2}(1-L)} = \frac{1}{w^{-2}} \\ &\Longrightarrow w^{-1}L + rKw^{-2} = \phi^{-2} - \phi^{-2}L \\ &\Longrightarrow \left(\phi^{-2} + w^{-1}\right) L = \phi^{-2} - rKw^{-2} \\ &\Longrightarrow L = \frac{\phi^{-2} - rKw^{-2}}{\phi^{-2} + w^{-1}} = \frac{\phi^{-2}w^2 - rK}{\phi^{-2}w^2 + w} = \frac{w^2 - rK\phi^2}{w^2 + w\phi^2} = \frac{w - \phi^2\left(\frac{r}{w}\right)K}{\phi^2 + w}. \end{aligned}$$

- Solution Method III: Optimality Condition Approach

Define: $r = \dfrac{R}{P}$ and $w = \dfrac{W}{P}$.

$$\max_{C,L} \ \sqrt{C} + \phi\sqrt{1-L} \quad \text{s.t.} \quad C = wL + rK$$
$$\Longrightarrow \max_{C,L} \ C^{\frac{1}{2}} + \phi\,(1-L)^{\frac{1}{2}} \quad \text{s.t.} \quad C = wL + rK$$

After transforming the constraint into real variable terms, $C = wL + rK$ indicates the **price of good is 1** and the **price of labor is w**.

Optimality condition: **ratio of marginal utility = price ratio**.

$$\frac{MU_C}{MU_L} = \frac{1}{w}$$
$$\Longrightarrow \frac{\frac{1}{2}C^{-\frac{1}{2}}}{\frac{1}{2}\phi(1-L)^{-\frac{1}{2}}(-1)} = \frac{1}{w}$$
$$\Longrightarrow \frac{C}{\phi^{-2}(1-L)} = \frac{1}{w^{-2}}$$

Substitue C with constraint: $$\Longrightarrow \frac{wL + rK}{\phi^{-2}(1-L)} = \frac{1}{w^{-2}}$$
$$\Longrightarrow w^{-1}L + rKw^{-2} = \phi^{-2} - \phi^{-2}L$$
$$\Longrightarrow \left(\phi^{-2} + w^{-1}\right)L = \phi^{-2} - rKw^{-2}$$
$$\Longrightarrow L = \frac{\phi^{-2} - rKw^{-2}}{\phi^{-2} + w^{-1}} = \frac{\phi^{-2}w^2 - rK}{\phi^{-2}w^2 + w} = \frac{w^2 - rK\phi^2}{w^2 + w\phi^2} = \frac{w - \phi^2\left(\frac{r}{w}\right)K}{\phi^2 + w}.$$