Midterm Review

1 Logistics

- Midterm exam **next** week (Nov. 9 8:20 pm to Nov. 10 8:20 pm)
- Best preparation materials: class notes, homework problems (covered in discussion sections).
- Handouts: http://www.haochehsu.com (Handout can be found at the *Teaching* section)
- Any **comments** to us, please feel free to use the anonymous *Feedback Survey*.
- We will go over **all** questions in the homework during discussion.

2 Overview of the Basic Consumer Theory

- Economics: The study of how to allocate scarce resources.
- Consumer theory: People have preferences and they optimize given the budget sets.
- Budgets sets describe the monetary resources one have. (Ch.2)
- Preferences describe the rankings the options. Without such rankings, it is not possible to allocate resources rationally. (Ch.3)
- Utility functions are another representations of the rankings. With utility functions, one can use calculus tools to analyze the model quantitatively. (Ch.4)
- (Optimal) choices are the results of the allocation process. Consumer choices are determined in a way that people choosing the best bundles that they can afford. (Ch.5)
- Demands summarize the optimal choices given prices and incomes. With demand functions, one can analyzes questions such as how does price affect people's optimal choices. (Ch.6)

3 Budget Sets

Definition 1. *A **budget set** B is a subset of choice set X that is defined by*

$$B(p_1, \ldots, p_n, M) := \{x \in X : p_1x_1 + \cdots + p_nx_n \leq M\}.$$

The budget line is the set of bundles that one spends all the money:

$$\{x \in X : p_1x_1 + \cdots + p_nx_n = M\},$$

In $n = 2$ case, budget constraint is a just a line. Recall the budget line is

$$p_1x_1 + p_2x_2 = M.$$

One can rewrite the above equation to obtain a linear function that is easy to draw on a graph:

$$x_2 = \frac{M}{p_2} - \frac{p_1}{p_2}x_1.$$

Exercises

Howard has just received a $\$1,000$ paycheck. His two favorite goods are video games and movie tickets. Video games are $\$50$ each and Movie tickets are $\$10$ each.

1. Write down the equation for Howard's budget constraint.
2. Graph Howard's budget constraint with video games on the X-axis and movie tickets on the Y-axis. Show the budget set on your graph.
3. What is the slope of Howard's budget constraint? What are the intercepts?
4. What happens to the budget line if the prices of both goods double?
5. What happens to the budget line if Howard's paycheck was $500 instead? Use the original prices for video games and movies.

4 Preferences

Definition 2. *A preference relation $\succsim$ on the set of alternatives X is a (binary) relation on X such that*

1. *the preference relation is complete, i.e., for all $x, y \in X$, either $x \succsim y$ or $y \succsim x$ or both cases happen;*
2. *the preference relation is reflexive,i.e., for all $x \in X$, $x \succsim x$.*
3. *the preference relation is transitive,i.e., for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.*

The symbol $\succsim$ can be interpreted as weakly preferred to. For example, $x \succsim y$ can be translated to x is weakly preferred to y. By the completeness assumption, given a preference relation $\succsim$ on X, we can say that

1. x is **strictly preferred** to y, write $x \succ y$, if $x \succsim y$ and $y \not\succsim x$. (translation: x is strictly preferred to y if x is weakly preferred to y and y is not weakly preferred to x).
2. x is **as preferred as** y, write $x \sim y$, if $x \succsim y$ and $y \succsim x$. (translation: x is as preferred as y if x is weakly preferred to y and y is weakly preferred to x).

- Well-behaved preference are monotonic (more is better) and convex (averages are preferred to extremes).

Definition 3. *For $n = 2$, a preference $\succsim$ on $X = \mathbb{R}^2$ is* ***monotonic*** *if for all (x_1, x_2) in X, $(x_1', x_2) \succsim (x_1, x_2)$ whenever $x_1' \geq x_1$ and $(x_1, x_2') \succsim (x_1, x_2)$ whenever $x_2' \geq x_2$.*

Definition 4. *A preference $\succsim$ is* ***convex*** *if for any $t \in (0, 1)$ and for any $x, y \in X$ such that $x \succsim y$, it is true that $tx + (1-t)y \succsim y$.*

Definition 5. *A preference $\succsim$ is* ***strictly convex*** *if for any $t \in (0, 1)$ and for any $x, y \in X$ such that $x \succsim y$ and $x \neq y$, it is true that $tx + (1-t)y \succ y$.*

Definition 6. *An* ***Indifference Curve*** *graphically describes all consumption bundles that the consumer is indifferent between.*

- ICs do not intersect.

Definition 7. *The **Marginal Rate of Substitution (MRS)** is the slope of a given indifference curve.*

- It describes how many units of x_2 one is willing to sacrifice (or be compensated, depending on whether the commodities are goods or not) in order to obtain an additional one unit of x_1.
- If we assume the two commodities are goods, then the slope of the IC is negative, because if you are losing one of the units, you will require some more units of the other goods to be as happy as before.
- Suppose that $x_2(x_1)$ is a function that describes an IC. Then, the MRS at (x_1, x_2) is MRS $= \frac{dx_2}{dx_1}(x_1)$.

Proposition 1. *Suppose a preference relation is monotonic. Then diminishing MRS (in absolute value) is equivalent to strictly convex preference.*

5 Utility Functions

Definition 8. *A utility function $u : X \to \mathbb{R}$ that represents a preference $\succsim$ if $u(x) \geq u(y)$ is equivalent to $x \succsim y$.*

- An important theorem tells that every 'nice' preference can be represented by some continuous utility function.
- In terms of utility function, the indifference curve now is the set of bundles that leads to the same utility level:
$$\text{IC} = \{(x_1, x_2) \in \mathbb{R}^2 : u(x_1, x_2) = c\}.$$
- For example, suppose $u(x_1, x_2) = x_1 x_2$. Then the indifference curve that gives utility level c is the graph of the function $x_2 = \frac{c}{x_1}$.
- One important property of the utility function is an increasing function a utility function represents the same preference.

Proposition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function (or monotonic transformation), i.e., for all x, y with $x \geq y$, then $f(x) \geq f(y)$. If u represents a preference $\succsim$ on X, then the function $w(x) = f\big(u(x)\big)$ is also a utility function that represents $\succsim$.*

Proof. (Optional) Suppose $x \succsim y$. Since u is a utility function that represents $\succsim$, we have $u(x) \geq u(y)$. We want to show that $w(x) \geq w(y)$. Indeed,

$$w(x) = f\big(u(x)\big) \geq f\big(u(y)\big) = w(y),$$

where the inequality is by the fact that f is an increasing function.
Now suppose $w(x) \geq w(y)$. We want to show that $x \succsim y$. By definition of w, we have

$$f\big(u(x)\big) = w(x) \geq w(y) = f\big(u(y)\big).$$

Since f is an increasing function, it follows that $u(x) \geq u(y)$. Because u represents $\succsim$, we must have $x \succsim y$. ■

- With calculus tools, we can also calculate the MRS by using utility function.
- Suppose you have some consumption bundle x, and we change it a bit, by $dx = (dx_1, dx_2)$.

- The change in utility (total derivatives) will be

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

- To stay on the same indifference curve, we can let $du = 0$. Then,

$$\frac{\partial u}{\partial x_1} dx_1 = -\frac{\partial u}{\partial x_2} dx_2.$$

  Equivalently, we have

$$\text{MRS} = \frac{dx_2}{dx_1} = -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = -\frac{\text{MU}_1}{\text{MU}_2},$$

  where we call the partial derivative as the marginal utility (MU).

- Here, if two commodities are two *goods*, then both marginal utilities are positive and therefore, the slope of indifference curve is negative, as we mentioned above.

- A monotonic transformation of utility will not change the MRS. Suppose f is an monotonic transformation of u. Let $w(x) = f\big(u(x)\big)$ be the utility function after transformation. By chain rule,

$$\frac{\partial w}{\partial x_1} = \frac{dw}{du} \cdot \frac{\partial u}{\partial x_1}.$$

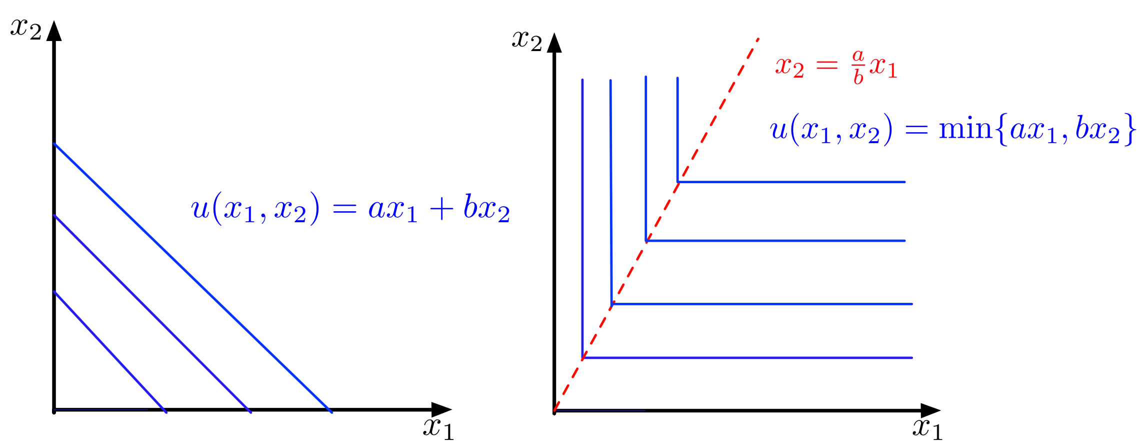
  By a similar calculation, we can obtain a similar equation for x_2. Therefore, the MRS in this case is

$$\text{MRS} = -\frac{\frac{\partial w}{\partial x_1}}{\frac{\partial w}{\partial x_2}} = -\frac{\frac{dw}{du} \cdot \frac{\partial u}{\partial x_1}}{\frac{dw}{du} \cdot \frac{\partial u}{\partial x_2}} = -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}},$$

  which is the same as before.

- The following three preferences are convex and monotonic.

  1. Perfect Substitutes:
     Utility function is $u(x_1, x_2) = ax_1 + bx_2$ for strictly positive a, b. Indifference curves are shown in figure 1a



(a) Indifference curves of perfect substitutes

(b) Indifference curves of perfect complements

Figure 1: figure 1a plots the indifference curves of perfect complements and figure 1b plots the indifference curves of perfect complements

2. Perfect Complements: Utility function is $u(x_1, x_2) = \min\{ax_1, bx_2\}$, for strictly positive a, b. Indifference curves are shown in figure 1b on the previous page

3. Cobb-Douglass: Utility function is $u(x_1, x_2) = x_1^a x_2^b$, for strictly positive a, b.Indifference curves are shown in figure 2

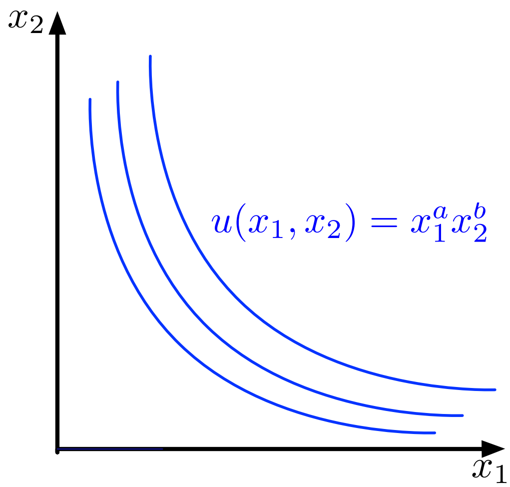


Figure 2: Indifference curves of Cobb-Douglass utility function

Exercises

There are two kinds of toppings available for pizza: mushrooms (M) and salami (S). Putting mushrooms on the horizontal axis and salami on the vertical axis, illustrate the following preferences by plotting indifference curves. Are preferences (strictly) monotonic? Are preferences (strictly) convex?

1. Utility from pizza can be described as $U(M, S) = \sqrt{MS}$. What is the marginal rate of substitution?

2. You only like pizza if it has both mushrooms and salami, mixed in proportion 2:1.

3. You are indifferent between having mushrooms or salami on your pizza. What is the marginal

4. You like salami and you don't care whether there are mushrooms on your pizza. What is the marginal rate of substitution? rate of substitution?

6 Choices

- For a preference relation on $\succsim$ on $\mathbb{R}^2$ and positive numbers p_1, p_2 and M, the consumer's problem is the problem of finding the best bundle in the budge set $B(p_1, p_2, M)$ according to $\succsim$. If $\succsim$ is represented by the utility function u, then this problem is

$$\max_{(x_1,x_2)\in X} u(x_1, x_2) \quad \text{subject to} \quad p_1 x_x + p_2 x_2 \leq M.$$

- With additional 'nice' assumptions about the preferences/utility functions, we can obtain the following facts:

Proposition 3. *Fixed a preference relation on $\mathbb{R}^2$ and a budget set.*

1. *If the utility function is continuous, then the consumer's problem has a solution.*
2. *If the preference relation is **strictly** convex, then the consumer's problem has at most one solution.*
3. *If the preference relation is monotonic, then any solution of consumer's problem is on the budget line.*

- For most preferences we will encounter in this course, they will be convex and monotonic.
- All the results till the end of section 6 assumes convex and monotonic preferences.
- Let $x_1^*(p_1, p_2, M)$ and $x_2^*(p_1, p_2, M)$ denote the optimal choices to the consumer's problem.
- We call (x_1^*, x_2^*) an interior solution if $x_1^* > 0$ and $x_2^* > 0$.
- If one of the choices is 0, then we call (x_1^*, x_2^*) a corner/boundary solution.
- Characterization of optimum: if (x_1^*, x_2^*) is an interior solution, then

$$\frac{\text{MU}_1}{p_1} = \frac{\text{MU}_2}{p_2} \qquad \textbf{(Bang per Buck} \text{ condition)} \tag{1}$$

- The ratio means "how much more utility I can buy for each dollar" if I invest in good i. At the optimum, both should be equal, otherwise you could take the last dollar you spent on one good, instead spend it on the other good, and make yourself better off.
- If optimally x_2^* is zero, then one must have,

$$\frac{\text{MU}_1}{p_1} \geq \frac{\text{MU}_2}{p_2}$$

- It means at $x_2^* = 0$, spending a dollar on goods 1 still marginally gives me more utility, however, I cannot trade good 2 for good 1 since I have already spent all the income on good 1.

7 Demands

- "Comparative statics" refers to changes in outcomes (such as demand) when parameters change (such as price or wealth).
- The price offer curve is the trajectory of (x_1^*, x_2^*) by changing one of the prices.
- The demand curve of good i is the function of $x_i^*(p_i)$ given the income and other prices fixed.
- The income offer curve is the trajectory of (x_1^*, x_2^*) by changing level of income.
- The Engel curve of good i is the function of $x_i^*(M)$ given all the prices fixed.
- If $x_i^*(p_1, p_2, M)$ is increasing in M, then we say good i is a **normal good**.
- If $x_i^*(p_1, p_2, M)$ is dereasing in M, then we say good i is a **inferior good**.
- A preference is **homothetic** if $(x_1, x_2) \succsim (y_1, y_2)$, then $(tx_1, tx_2) \succsim (ty_1, ty_2)$ for all $t > 0$.
- If a preference is homothetic, then for any two bundles that have the same ratio, they will have the same MRS. (To prove it rigorous, it requires some sophisticated results, which is not covered in the class. You should take it as granted. A graphical illustration is shown in figure 3 on the next page)
- The Engel curve of homothetic preference is **linear**.

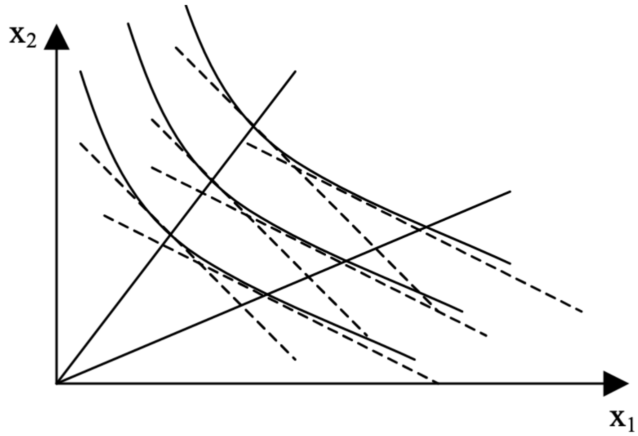


Figure 3: Slope of the IC only depends on the ratio of the goods.

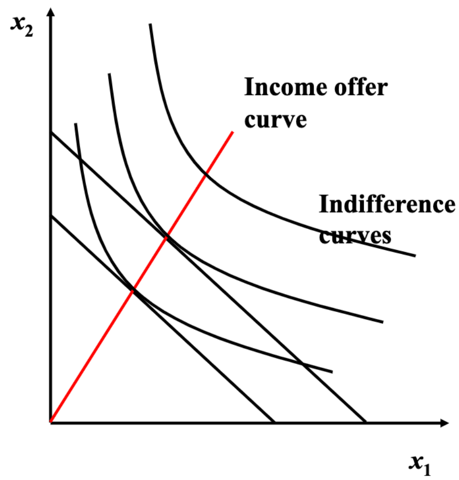


Figure 4: The Engel curve of homoethetic preference is linear.

Exercises

Answer the questions below for each of the following preferences:

(a) $U(x, y) = \min\{4x, y\}$

(b) $U(x, y) = x + 4y$

(c) $U(x, y) = xy^2$

For each question, complete the required calculations **and** give an economic interpretation of your results. In each case, set $p_1 = 2, p_2 = 1, M = 12$.

1. Suggest an example of goods that would fit the described preferences.
2. Is this preference homothetic?
3. Derive the demand for each good.
4. Derive and plot the income offer curve and the Engel curve for good x.
5. Derive and plot the price offer curve for good x.
6. Plot a demand curve for good x.