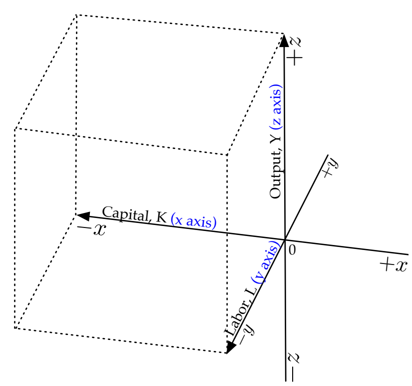
Discussion 3

1 The Production Function

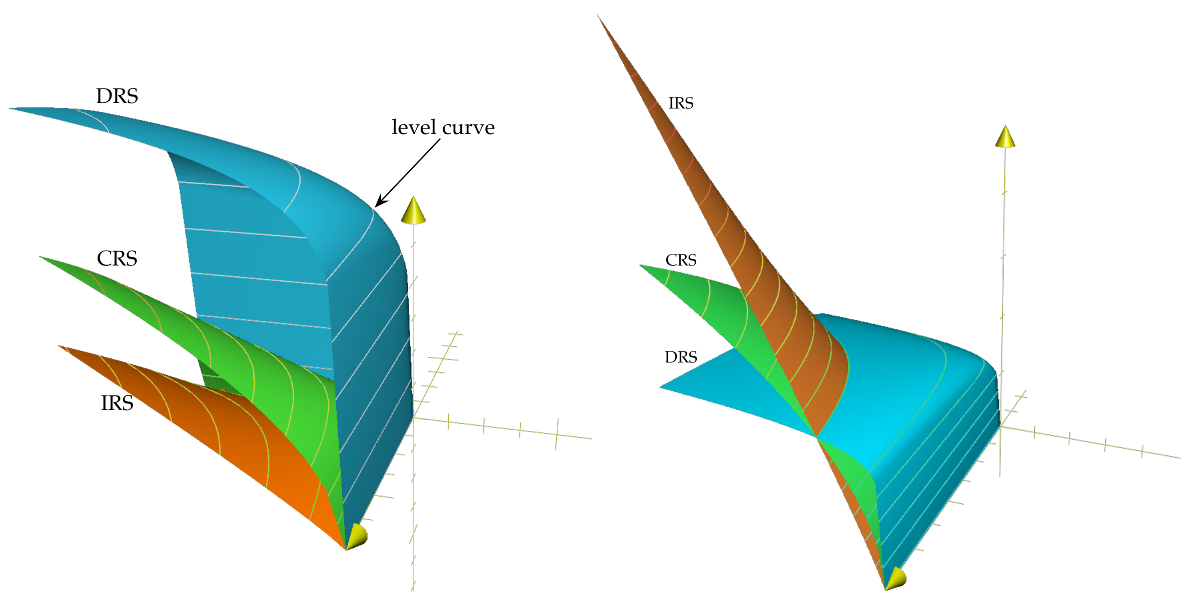
$$Y = A \cdot F(L, K, H, N) \tag{1}$$

- Describe the technology of a firm and gives the **maximum** output associated with inputs.
- Vector representation: *production plan = (Y, -L, -K, -H, -N)*
  - Positive factor: Output
  - Negative factor: Input
- The components:



  1. Y: quantity of **output**
  2. A: stock of technological **knowledge**
  3. F: transformation function
  4. L: quantity of **labor** (number of workers)
  5. K: stock of physical capital or "**capital**"
  6. H: stock of **human capital**
  7. N: stock of **natural resources**

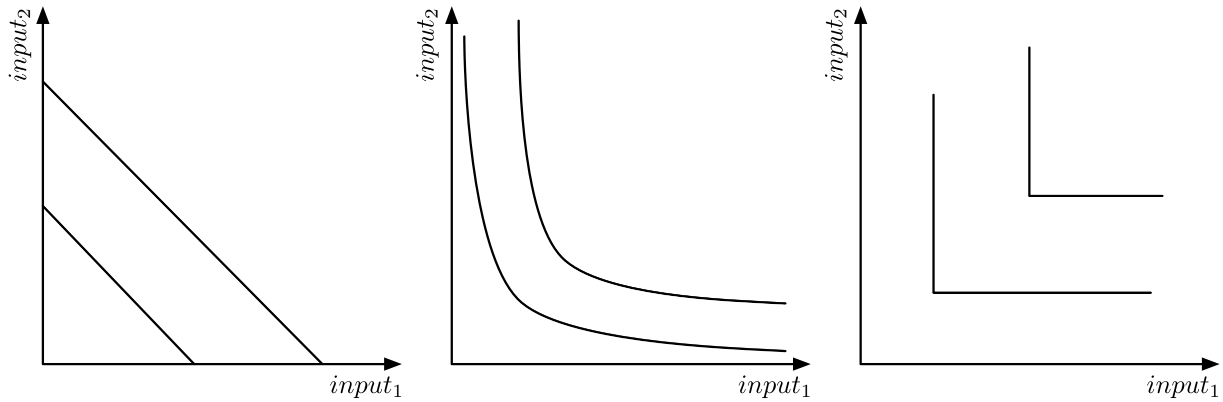
- Visualization: suppose we use Capital(K) and Labor(L) to produce with a Cobb-Douglas[1] technology.



The graphs represent three different function characteristics: Increasing returns to scale(IRS), constant return to scale(CRS), decreasing return to scale(DRS).

[1] The Cobb-Douglas technology produces a single good using input factors with the function form $Y = A \cdot L^{\alpha} K^{\beta}$ where $\alpha, \beta \in (0, 1)$.

If we slice it horizontally (looking downwards directly from above), the level curves on a two-dimensional plane illustrate the isoquant[2] curve of the Cobb-Douglas technology in the middle graph below.



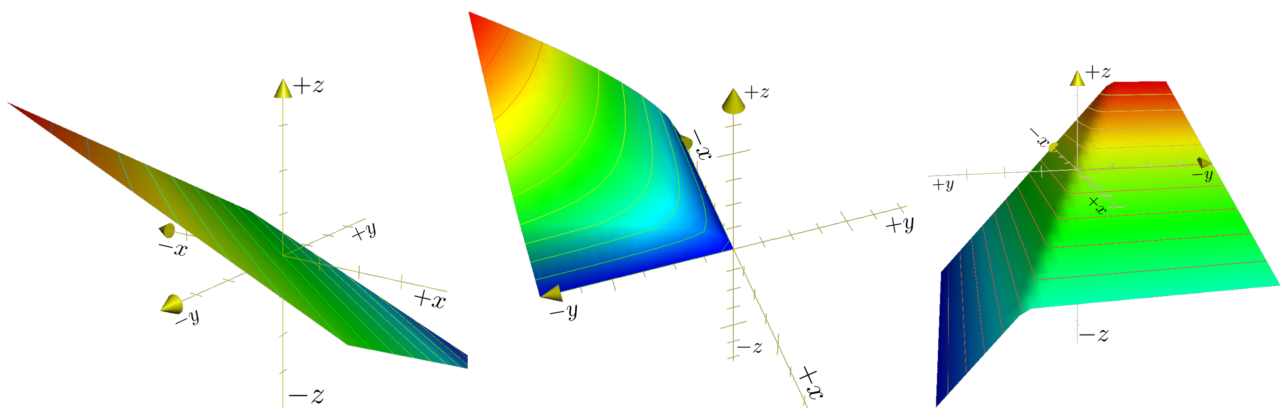
Linear:
$\text{Y} = a(input_1) + b(input_2)$

Cobb-Douglas:
$\text{Y} = (input_1)^{\alpha} \cdot (input_2)^{\beta}$

Leontief:
$\text{Y} = \min\left\{a(input_1), b(input_2)\right\}$

If we change our view angle from overlook to a more horizontal point of view, then we have the following figures. The color gradient represents the height of the graph. The contour lines illustrate the level curve.

Notice that the part of the graph with a positive z (the part of the graph that is above the 0 horizontal plane) is always in the third quadrant. Link with section one, think of $-x$ and $-y$ as two "inputs ". When the machine "uses "the inputs, their quantity decreases. This production characteristic is captured by the negative sign. As for the positive z, it is the output from the production. In a sense that when we produce, we create new products. Hence, the output quantity increases.



The 3D visualization from left to right represents Linear, Cobb-Douglas, and Leontief.

2 Returns to scale

Recall that the production function is a "machine"that turns several different inputs to some output. What will happen to the *level of output* when we give the machine *double*, *triple* or even **t** times of inputs?

1. Increasing returns to scale: a technology has IRS: $F(\mathbf{t} \cdot input_1, \dots, \mathbf{t} \cdot input_n) > \mathbf{t} \cdot F(input_1, \dots, input_n)$
2. Constant returns to scale: a technology has CRS: $F(\mathbf{t} \cdot input_1, \dots, \mathbf{t} \cdot input_n) = \mathbf{t} \cdot F(input_1, \dots, input_n)$
3. Decreasing returns to scale: a technology has DRS: $F(\mathbf{t} \cdot input_1, \dots, \mathbf{t} \cdot input_n) < \mathbf{t} \cdot F(input_1, \dots, input_n)$

[2] An isoquant is a curve that shows all the combinations of inputs that yield the same level of output.

3 Productivity

We will derive the level of productivity from equation 1. Noticed that there is an essential assumption regarding the production function. We assume that holding the stock of technological knowledge (A) fixed, the production function exhibit constant returns to scale, CRS, i.e. the production function is homogeneous of degree 1. Then we can rewrite the production function

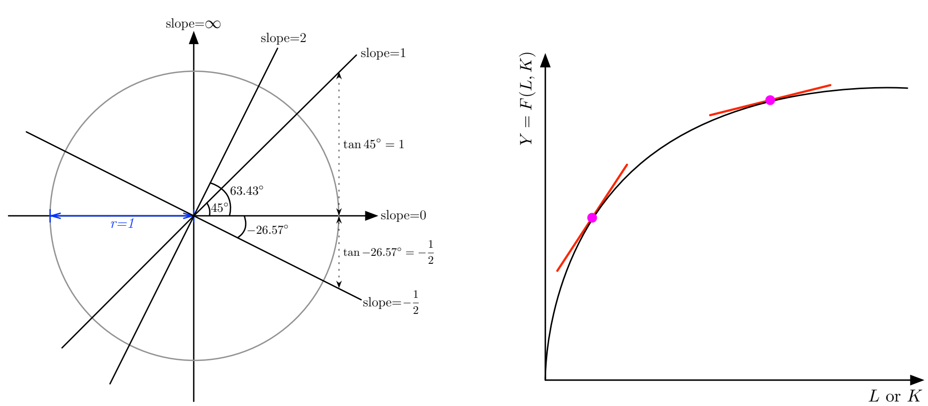
$$
\begin{aligned}
Y &= A \cdot F(L, K, H, N) \\
&= A \cdot L \cdot F\left(\frac{L}{L}, \frac{K}{L}, \frac{H}{L}, \frac{N}{L}\right) \\
&= AL \cdot F\left(1, \frac{K}{L}, \frac{H}{L}, \frac{N}{L}\right) \\
\xrightarrow{\text{divide both sides by L}} \frac{Y}{L} &= A \cdot F\left(1, \frac{K}{L}, \frac{H}{L}, \frac{N}{L}\right).
\end{aligned}
$$

Here we define the output (Y) per worker: $\frac{Y}{L}$ as *productivity*. Interpret directly from the production function, we can conclude that the productivity is determined by:

1. technological knowledge (L↑ $\Longrightarrow A \uparrow$)
2. capital per worker (L↑ $\Longrightarrow \frac{K}{L} \downarrow$)
3. human capital per worker (L↑ $\Longrightarrow \frac{H}{L} \downarrow$)
4. natural resources per worker (L↑ $\Longrightarrow \frac{N}{L} \downarrow$)
   (a) A↑$\Longrightarrow$ production process become more resource-efficient.
   (b) Since natural resources is limited, the supply of resources is decreasing but the demand is decreasing just as fast or even more rapidly.
   (c) In conclusion, the limitation of natural resources won't become a significant barrier that limits economic growth.

4 Diminishing Marginal Returns

1. Diminishing returns refers to property whereby the benefit from an extra unit of input declines as the quantity of the input increases.
2. Higher savings leads in the long run to higher levels of productivity and incomes but not to higher growth rates in these variables.
3. *Catch-up Effect:* countries that start off poor tend to grow more rapidly than countries that start off rich.



Angle and slope on the unit circle(left), Concave production function(right)

5 Investment from Abroad

- A country's capital stock can also be augmented by investment from abroad:
  1. *Foreign direct investment* occurs when foreigners make capital investments that they own and operate in the domestic economy.
  2. *Foreign portfolio investment* occurs when foreigners lend money to domestic corporations that use the funds to acquire more physical capital.
- When foreigners invest in a country, they expect to earn a return. But the capital they supply makes domestic workers more productive, increasing the workers' incomes as well.
- The World Bank raises funds in advanced countries and uses those funds to make loans in developing countries.

6 Exercises

1. Which of the following is physical capital?
   (a) The strength of workers.
   (b) The equipment in a factory.
   (c) Financial assets like cash and bonds.
   (d) The knowledge of workers.
2. Which of the following is a characteristic that is not generally shared by countries with higher levels of GDP per capita:
   (a) longer life expectancy.
   (b) more physical capital per worker.
   (c) lower rates of child mortality.
   (d) lower productivity.
3. Sabrina is a landscaper. Which of the following are included in her human capital?
   (a) Neither her knowledge of landscaping learned in college nor her landscaping equipment
   (b) Her knowledge of landscaping learned in college, but not her landscaping equipment
   (c) Her landscaping equipment, but not her knowledge of landscaping learned in college
   (d) Her knowledge of landscaping learned in college and her landscaping equipment
4. Other things equal, which of the following would increase productivity?
   (a) An increase in either human or physical capital.
   (b) An increase in physical capital but not an increase in human capital.
   (c) An increase in human capital but not an increase in physical capital.
   (d) Neither an increase in human capital nor an increase in physical capital.
5. Productivity is:
   (a) the amount of time that the average worker in a country spends working.
   (b) the cost that the average worker incurs to produce a given quantity of goods.
   (c) the amount of knowledge that the average worker possesses.
   (d) the quantity of goods and services that the average worker in a country produces in a given period of time.

6. Alex is a plumber. Which of the following are included in his physical capital?
   (a) The tools he uses, but not the knowledge he learned on the job.
   (b) The knowledge he learned on the job, and the tools he uses.
   (c) The knowledge he learned on the job, but not the tools he uses.
   (d) Neither the knowledge he learned on the job nor the tools he uses.
7. The inputs into production of goods and services that are provided by nature, such as land, rivers, and mineral deposits are called:
   (a) technological knowledge.
   (b) physical capital.
   (c) human capital.
   (d) natural resources.
8. The production process –e.g., the steps required, the ingredients and tools needed –for manufacturing a drug for treating high blood pressure is an example of:
   (a) human capital.
   (b) physical capital.
   (c) natural resources.
   (d) technological knowledge.
9. In the 1800s, European residents purchased stock in American companies that then used the funds to build railroads and factories. The Europeans who did this were engaging in:
   (a) foreign indirect investment.
   (b) foreign direct investment.
   (c) indirect domestic investment.
   (d) foreign portfolio investment.
10. Which of the following characteristics best illustrates counties that generates a catch-up effect?
   (a) countries with lower productivity growth and less capital per person
   (b) countries with higher productivity growth and less capital per person
   (c) countries with higher productivity growth and more capital per person
   (d) countries with higher population growth and more capital per person

7 Appendix

Here we will introduce the general characterization of returns to scale.

7.1 Homogeneous Functions

Definition 1 (Homogeneous Functions)**.** *A real value function $F(tx_1, \dots, tx_n)$ is homogeneous of degree $\boldsymbol{k}$ if $\forall t > 0$*

$$F(tx_1, \dots, tx_{\boldsymbol{n}}) = t^{\boldsymbol{k}} \cdot F(x_1, \dots, x_{\boldsymbol{n}}). \tag{2}$$

7.2 Returns to scale

1. Increasing returns to scale: the production function is homogeneous of degree $k > 1$.
2. Constant returns to scale: the production function is homogeneous of degree $k = 1$.
3. Decreasing returns to scale: the production function is homogeneous of degree $k < 1$.

7.3 Example

Suppose a Cobb-Douglas production function takes the form

$$A \cdot F(x_1, x_2, \dots, x_n) = A x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdots \cdot x_n^{\alpha_n}.$$

Then this function is homogeneous of degree $k = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Hence, if $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then this function exhibits constant returns to scale.

7.4 Application of the function degree

Theorem 1 (Euler's Theorem)**.** *Let $F(x_1, \dots, x_n)$ be a function that is homogeneous of degree $\boldsymbol{k}$. Then*

$$x_1 \frac{\partial F}{\partial x_1}(\boldsymbol{x}) + \cdots + x_n \frac{\partial F}{\partial x_n}(\boldsymbol{x}) = \boldsymbol{k} \cdot F(\boldsymbol{x}). \tag{3}$$

With the gradient notation, we can rewrite the equation to

$$\boldsymbol{x} \cdot \nabla F(\boldsymbol{x}) = \boldsymbol{k} \cdot F(\boldsymbol{x}). \tag{4}$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$.